

# The Fundamental Welfare Theorems

The so-called Fundamental Welfare Theorems of Economics tell us about the relation between market equilibrium and Pareto efficiency.

**The First Welfare Theorem:** Every Walrasian equilibrium allocation is Pareto efficient.

**The Second Welfare Theorem:** Every Pareto efficient allocation can be supported as a Walrasian equilibrium.

The theorems are certainly not true in the unconditional form in which we've stated them here. A better way to think of them is this: "Under certain conditions, a market equilibrium is efficient" and "Under certain conditions, an efficient allocation can be supported as a market equilibrium." Nevertheless, these two theorems really are fundamental benchmarks in microeconomics. We're going to give conditions under which they're true. One (very stringent) set of conditions will enable us to prove the theorems with the calculus methods (*i.e.*, Kuhn-Tucker, Lagrangian, gradient methods) we've used so fruitfully in our analysis of Pareto efficiency. Additionally, for each theorem we'll provide a much weaker set of conditions under which the theorem remains true.

Assume to begin with, then, that all the consumers' preferences are "very nice" — *viz.*, that they're representable by utility functions  $u^i$  that satisfy the following condition:

$$u^i \text{ is continuously differentiable, strictly quasiconcave, and } \forall \mathbf{x}^i \in \mathbb{R}_+^l : u_k^i(\mathbf{x}^i) > 0 \quad (*)$$

**The First Welfare Theorem:** If  $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium for an economy  $E = ((u^i, \hat{\mathbf{x}}^i)_1^n)$  in which each  $u^i$  satisfies  $(*)$ , then  $(\hat{\mathbf{x}}^i)_1^n$  is a Pareto allocation for  $E$ .

**Proof:**

Because  $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium for  $E$ , each  $\hat{\mathbf{x}}^i$  maximizes  $u^i$  subject to  $\mathbf{x}^i \geq \mathbf{0}$  and to the budget constraint  $\hat{\mathbf{p}} \cdot \mathbf{x}^i \leq \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i$ . Therefore, for each  $i \in N$  there is a  $\lambda_i \geq 0$  that satisfies the first-order marginal conditions for  $i$ 's maximization problem:

$$\forall k : u_k^i \leq \lambda_i \hat{p}_k, \text{ with equality if } \hat{x}_k^i > 0, \quad (1)$$

where of course the partial derivatives  $u_k^i$  are evaluated at  $\hat{\mathbf{x}}^i$ . In fact, because each  $u_k^i > 0$ , we have  $\lambda_i > 0$  and  $\hat{p}_k > 0$  for each  $i$  and each  $k$ . The market-clearing condition in the definition of Walrasian equilibrium therefore yields

$$\sum_{i=1}^n \hat{x}_k^i = \sum_{i=1}^n \hat{x}_k^i \text{ for each } k \text{ (because each } \hat{p}_k > 0). \quad (2)$$

For each  $i$  we can define  $\lambda'_i = \frac{1}{u_k^i}$ , because  $\lambda_i > 0$ , and then we can rewrite (1) as

$$\forall k : \lambda'_i u_k^i \leq \widehat{p}_k, \text{ with equality if } \widehat{x}_k^i > 0, \quad (3)$$

for each  $i \in N$ . But (2) and (3) are exactly the first-order conditions that characterize the solutions of the Pareto maximization problem (P-max). Therefore  $(\widehat{\mathbf{x}}^i)_1^n$  is a solution of (P-max), and since each  $u_k^i > 0$ , a solution of (P-max) is a Pareto allocation. ■

In the proof, we could have instead expressed the first-order marginal conditions for the individual consumers' maximization problems in terms of marginal rates of substitution:

$$\forall k, k' : MRS_{kk'}^i = \frac{\widehat{p}_k}{\widehat{p}_{k'}} \quad (4)$$

(written for the interior case, to avoid a lot of inequalities), which yields the (Equal MRS) condition

$$\forall i, j, k, k' : MRS_{kk'}^i = MRS_{kk'}^j. \quad (5)$$

The conditions (5) and (2) then guarantee that  $(\widehat{\mathbf{x}}^i)_1^n$  is a Pareto allocation.

Now let's see if we can prove the First Welfare Theorem without any of the assumptions in (\*). It turns out that we can't quite do that; however, the only thing we need to assume is that each consumer's preference  $\succsim_i$  is locally nonsatiated (LNS).

**The First Welfare Theorem:** If  $(\widehat{\mathbf{p}}, (\widehat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium for an economy  $E = ((\succsim_i, \widehat{\mathbf{x}}^i)_1^n)$  in which each  $\succsim_i$  is locally nonsatiated, then  $(\widehat{\mathbf{x}}^i)_1^n$  is a Pareto allocation for  $E$ .

**Proof:**

Suppose  $(\widehat{\mathbf{x}}^i)_1^n$  is not a Pareto allocation — *i.e.*, some allocation  $(\widetilde{\mathbf{x}}^i)_1^n$  is a Pareto improvement on  $(\widehat{\mathbf{x}}^i)_1^n$ :

$$\begin{aligned} (a) \quad & \sum_1^n \widetilde{\mathbf{x}}^i \leq \sum_1^n \widehat{\mathbf{x}}^i \\ (b1) \quad & \forall i \in N : \widetilde{\mathbf{x}}^i \succsim_i \widehat{\mathbf{x}}^i \\ (b2) \quad & \exists i \in N : \widetilde{\mathbf{x}}^i \succ_i \widehat{\mathbf{x}}^i. \end{aligned}$$

Because  $(\widehat{\mathbf{p}}, (\widehat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium for  $E$ , each  $\widehat{\mathbf{x}}^i$  is maximal for  $\succsim_i$  on the budget set  $\mathcal{B}(\widehat{\mathbf{p}}, \widehat{\mathbf{x}}^i) := \{ \mathbf{x}^i \in \mathbb{R}_+^l \mid \widehat{\mathbf{p}} \cdot \mathbf{x}^i \leq \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^i \}$ . Therefore, (b2) implies that

$$\exists i \in N : \widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i > \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^i. \quad (6)$$

Moreover, since each  $\succsim_i$  is LNS and each  $\mathbf{x}^i$  is  $\succsim_i$ -maximal on  $\mathcal{B}(\widehat{\mathbf{p}}, \widehat{\mathbf{x}}^i)$ , (b1) implies that

$$\forall i \in N : \widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i \geq \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^i, \quad (7)$$

as follows: assume that  $\widetilde{\mathbf{x}}^i \succsim_i \widehat{\mathbf{x}}^i$  and suppose that  $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i < \widehat{\mathbf{p}} \cdot \widehat{\mathbf{x}}^i$ ; then, because  $\succsim_i$  is LNS, there is a bundle  $\mathbf{x}$  that satisfies both  $\widehat{\mathbf{p}} \cdot \widetilde{\mathbf{x}}^i < \widehat{\mathbf{p}} \cdot \mathbf{x}$  and  $\mathbf{x} \succ_i \widetilde{\mathbf{x}}^i$ , which contradicts the maximality of  $\widehat{\mathbf{x}}^i$  on the budget set  $\mathcal{B}(\widehat{\mathbf{p}}, \widehat{\mathbf{x}}^i)$ .

Summing the inequalities in (6) and (7) over all  $i \in N$  yields

$$\sum_{i=1}^n \hat{\mathbf{p}} \cdot \tilde{\mathbf{x}}^i > \sum_{i=1}^n \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \quad (8)$$

*i.e.*,

$$\hat{\mathbf{p}} \cdot \sum_{i=1}^n \tilde{\mathbf{x}}^i > \hat{\mathbf{p}} \cdot \sum_{i=1}^n \hat{\mathbf{x}}^i. \quad (9)$$

Since  $\hat{\mathbf{p}} \in \mathbb{R}_+^l$ , it follows from (9) that there is at least one good  $k$  for which

$$\sum_{i=1}^n \tilde{x}_k^i > \sum_{i=1}^n \hat{x}_k^i \quad (10)$$

— *i.e.*,  $(\tilde{\mathbf{x}}^i)_1^n$  does not satisfy (a). Our assumption that  $(\tilde{\mathbf{x}}^i)_1^n$  is a Pareto improvement has led to a contradiction; therefore there are no Pareto improvements on  $(\hat{\mathbf{x}}^i)_1^n$ , and  $(\hat{\mathbf{x}}^i)_1^n$  is therefore a Pareto allocation. ■

**The Second Welfare Theorem:** Let  $(\hat{\mathbf{x}}^i)_1^n$  be a Pareto allocation for an economy in which the utility functions  $u^1, \dots, u^n$  all satisfy (\*) and in which the total endowment of goods is  $\hat{\mathbf{x}} \in \mathbb{R}_{++}^l$ . Then there is a price-list  $\hat{\mathbf{p}} \in \mathbb{R}_+^l$  such that

for every  $(\hat{\mathbf{x}}^i)_1^n$  that satisfies

$$\sum_{i=1}^n \hat{\mathbf{x}}^i = \hat{\mathbf{x}} \quad \text{and} \quad \forall i : \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i = \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \quad (11)$$

$(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium of the economy  $E = ((u^i, \hat{\mathbf{x}}^i))_1^n$ .

— *i.e.*,  $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium of the economy in which each consumer has the utility function  $u^i$  and the initial bundle  $\hat{\mathbf{x}}^i$ .

**Proof:**

Let  $(\hat{\mathbf{x}}^i)_1^n$  be a Pareto allocation for the given utility functions and endowment  $\hat{\mathbf{x}}$ . We first show that the conclusion holds for  $(\hat{\mathbf{x}}^i)_1^n = (\hat{\mathbf{x}}^i)_1^n$  — *i.e.*, if each consumer's initial bundle is  $\hat{\mathbf{x}}^i$ . Since  $(\hat{\mathbf{x}}^i)_1^n$  is a Pareto allocation, it is a solution of the Pareto maximization problem (P-max), and it therefore satisfies the first-order marginal conditions

$$\forall i, k : \exists \lambda_i, \sigma_k \geq 0 : \lambda_i u_k^i \leq \sigma_k, \quad \text{with equality if } \hat{x}_k^i > 0. \quad (12)$$

Each  $\lambda_i$  is the Lagrange multiplier for one of the utility-level constraints in (P-max), and since  $u_k^i > 0$  for each  $i$  and each  $k$ , it's clear that each constraint's Lagrange multiplier must be positive: relaxing any one of the constraints will allow  $u^1$  to be increased.

For each  $k$ , let  $\hat{p}_k = \sigma_k$ . Now (12) yields, for each  $i$ ,

$$\exists \lambda_i : \forall k : u_k^i \leq \frac{1}{\lambda_i} \hat{p}_k, \quad \text{with equality if } \hat{x}_k^i > 0. \quad (13)$$

These are exactly the first-order marginal conditions for consumer  $i$ 's utility-maximization problem at the price-list  $\hat{\mathbf{p}}$ . Therefore, since each consumer's initial bundle is  $\hat{\mathbf{x}}^i$ , each consumer is maximizing  $u^i$  at  $\hat{\mathbf{x}}^i$ . And since  $\hat{p}_k = \sigma_k > 0$  for each  $k$ , the constraint satisfaction first-order condition for (P-Max) yields  $\sum_{i=1}^n \hat{\mathbf{x}}^i = \hat{\mathbf{x}}$ , so the market-clearing condition for a Walrasian equilibrium is satisfied as well. Thus, we've shown that all the conditions in the definition of Walrasian equilibrium are satisfied for  $(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$ .

If each consumer's initial bundle is some other  $\hat{\mathbf{x}}^i$  instead of  $\hat{\mathbf{x}}^i$ , but still satisfying (11), then the second equation in (11) ensures that each consumer's budget constraint is the same as before (the right-hand sides have the same value), and therefore the inequalities in (13) again guarantee that each  $\hat{\mathbf{x}}^i$  maximizes  $u^i$  subject to the consumer's budget constraint. And as before, the constraint satisfaction first-order condition for (P-Max),  $\sum_{i=1}^n \hat{\mathbf{x}}^i = \hat{\mathbf{x}}$ , coincides with the market-clearing condition for a Walrasian equilibrium. ■

Just as with the First Welfare Theorem, the Second Theorem is true under weaker assumptions than those in (\*) — but in the case of the Second Theorem, not that much weaker: we can dispense with the differentiability assumption, and we can weaken the convexity assumption and the assumption that utility functions are strictly increasing.

**The Second Welfare Theorem:** Let  $(\hat{\mathbf{x}}^i)_1^n$  be a Pareto allocation for an economy in which each  $u^i$  is continuous, quasiconcave, and locally nonsatiated, and in which the total endowment of goods is  $\hat{\mathbf{x}} \in \mathbb{R}_{++}^l$ . Then there is a price-list  $\hat{\mathbf{p}} \in \mathbb{R}_+^l$  such that

if the no-minimum-wealth condition

$$\forall i \in N : \exists \mathbf{x}^i \in \mathbb{R}_+^l : \hat{\mathbf{p}} \cdot \mathbf{x}^i < \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \quad (\text{NMW})$$

is satisfied, then for every  $(\hat{\mathbf{x}}^i)_1^n$  that satisfies

$$\sum_{i=1}^n \hat{\mathbf{x}}^i = \hat{\mathbf{x}} \quad \text{and} \quad \forall i : \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i = \hat{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \quad (14)$$

$(\hat{\mathbf{p}}, (\hat{\mathbf{x}}^i)_1^n)$  is a Walrasian equilibrium of the economy  $E = ((u^i, \hat{\mathbf{x}}^i))_1^n$ .

Note that the no-minimum-wealth condition (NMW) is satisfied if the much stronger condition that each  $\hat{\mathbf{x}}^i \in \mathbb{R}_{++}^l$  is satisfied, *i.e.*, that each  $\hat{x}_k^i > 0$ .

The proof of the Second Theorem at this level of generality is not nearly as straightforward as the proof of the First Theorem. The proof requires some investment in the convex analysis. We'll cover this proof after we've covered the necessary convex analysis in the Math Course.

## Duality Theorems in Demand Theory: Utility Maximization and Expenditure Minimization

These two Duality Theorems of demand theory tell us about the relation between utility maximization and expenditure minimization — *i.e.*, between Marshallian demand and Hicksian (or compensated) demand. We would like to know that the utility-maximization hypothesis ensures that any bundle a consumer chooses (if he is a price-taker) must minimize his expenditure over all the bundles that would have made him at least as well off. And conversely, we would like to know that a bundle that minimizes expenditure to attain a given utility level must maximize his utility among the bundles that don't cost more. The following two examples show that the two ideas are not always the same.

**Example 1:** A consumer with a thick indifference curve, as in Figure 1, where there are utility-maximizing bundles that do not minimize expenditure.

**Example 2:** A consumer whose wealth is so small that any reduction in it would leave him with no affordable consumption bundles, as in Figure 2, where there are expenditure-minimizing bundles that do not maximize utility.

The examples suggest assumptions that will rule out such situations.

### Utility Maximization Implies Expenditure Minimization

It's clear in Example 1 that the reason  $\hat{\mathbf{x}}$  fails to minimize  $\mathbf{p} \cdot \mathbf{x}$  is because the preference is not locally nonsatiated. If a consumer's preference *is* locally nonsatiated, then preference maximization does imply expenditure minimization.

**First Duality Theorem:** If  $\succsim$  is a locally nonsatiated preference on a set  $X$  of consumption bundles in  $\mathbb{R}_+^\ell$ , and if  $\hat{\mathbf{x}}$  is  $\succsim$ -maximal in the budget set  $\{x \in X \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \hat{\mathbf{x}}\}$ , then  $\hat{\mathbf{x}}$  minimizes  $\mathbf{p} \cdot \mathbf{x}$  over the upper-contour set  $\{x \in X \mid \mathbf{x} \succsim \hat{\mathbf{x}}\}$ .

**Proof:** (See Figure 3)

Suppose  $\hat{\mathbf{x}}$  does not minimize  $\mathbf{p} \cdot \mathbf{x}$  on  $\{x \in X \mid \mathbf{x} \succsim \hat{\mathbf{x}}\}$ . Then there is a bundle  $\tilde{\mathbf{x}} \in X$  such that  $\tilde{\mathbf{x}} \succ \hat{\mathbf{x}}$  and  $\mathbf{p} \cdot \tilde{\mathbf{x}} < \mathbf{p} \cdot \hat{\mathbf{x}}$ . Let  $\mathcal{N}$  be a neighborhood of  $\tilde{\mathbf{x}}$  for which  $\mathbf{x} \in \mathcal{N} \Rightarrow \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \hat{\mathbf{x}}$  — *i.e.*,  $\mathcal{N}$  lies entirely below the  $\mathbf{p}$ -hyperplane through  $\hat{\mathbf{x}}$ . Since  $\succsim$  is locally nonsatiated, there is a bundle  $\mathbf{x}' \in \mathcal{N}$  that satisfies  $\mathbf{x}' \succ \tilde{\mathbf{x}}$ , and *a fortiori*  $\mathbf{x}' \succ \hat{\mathbf{x}}$ . But then we have both  $\mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \hat{\mathbf{x}}$  and  $\mathbf{x}' \succ \hat{\mathbf{x}}$  — *i.e.*,  $\hat{\mathbf{x}}$  does not maximize  $\succsim$  on the budget set, a contradiction.

■

## Expenditure Minimization Implies Utility Maximization

It seems clear that the reason  $\hat{\mathbf{x}}$  doesn't maximize utility in Example 2 among the bundles costing no more than  $\hat{\mathbf{x}}$  is that there are no bundles in the consumer's consumption set that cost strictly less than  $\hat{\mathbf{x}}$ . If there were such a bundle, we could move "continuously" along a line from that bundle to any bundle  $\tilde{\mathbf{x}}$  costing no more than  $\hat{\mathbf{x}}$  but strictly better than  $\hat{\mathbf{x}}$ , and we would have to encounter a bundle that also costs strictly less than  $\hat{\mathbf{x}}$  and that also is strictly better than  $\hat{\mathbf{x}}$  — so that  $\hat{\mathbf{x}}$  would not maximize utility among the bundles costing no more than  $\hat{\mathbf{x}}$ . Note, though, that in addition to having some bundle that costs less than  $\hat{\mathbf{x}}$ , this argument also requires both continuity of the preference and convexity of the set of possible bundles. The proof of the Second Duality Theorem uses exactly this argument, so it requires these stronger assumptions, which were not needed for the First Duality Theorem. Can you produce counterexamples to show that the theorem does indeed require each of these assumptions — continuity and convexity — in addition to the counterexample in Figure 2 in which there is no cheaper bundle than  $\hat{\mathbf{x}}$  in the consumer's set of possible consumptions?

Since we have to assume the preference is continuous, it will therefore be representable by a continuous utility function, so we state and prove the theorem in terms of a continuous utility function. We could instead do the proof with a continuous preference  $\succsim$ .

**Second Duality Theorem:** Let  $u : X \rightarrow \mathbb{R}$  be a continuous function on a convex set  $X$  of consumption bundles in  $\mathbb{R}_+^\ell$ . If  $\hat{\mathbf{x}}$  minimizes  $\mathbf{p} \cdot \mathbf{x}$  over the upper-contour set  $\{x \in X \mid u(\mathbf{x}) \geq u(\hat{\mathbf{x}})\}$ , and if  $X$  contains a bundle  $\mathbf{x}$  that satisfies  $\mathbf{p} \cdot \mathbf{x} < \mathbf{p} \cdot \hat{\mathbf{x}}$ , then  $\hat{\mathbf{x}}$  maximizes  $u$  over the budget set  $\{x \in X \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \hat{\mathbf{x}}\}$ .

**Proof:** (See Figure 4)

Suppose that  $\hat{\mathbf{x}}$  minimizes  $\mathbf{p} \cdot \mathbf{x}$  over the set  $\{x \in X \mid u(\mathbf{x}) \geq u(\hat{\mathbf{x}})\}$ , but that  $\hat{\mathbf{x}}$  does not maximize  $u$  over the set  $\{x \in X \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \hat{\mathbf{x}}\}$  — *i.e.*, there is some  $\tilde{\mathbf{x}} \in X$  such that  $\mathbf{p} \cdot \tilde{\mathbf{x}} \leq \mathbf{p} \cdot \hat{\mathbf{x}}$  but  $u(\tilde{\mathbf{x}}) > u(\hat{\mathbf{x}})$ . Since  $u$  is continuous, there is a neighborhood  $\mathcal{N}$  of  $\tilde{\mathbf{x}}$  such that each  $\mathbf{x} \in \mathcal{N}$  also satisfies  $u(\mathbf{x}) > u(\hat{\mathbf{x}})$ . We've furthermore assumed that there is a bundle  $\mathbf{x}' \in X$  that satisfies  $\mathbf{p} \cdot \mathbf{x}' < \mathbf{p} \cdot \hat{\mathbf{x}}$ . Since  $X$  is convex, the neighborhood  $\mathcal{N}$  contains a convex combination, say  $\mathbf{x}''$ , of  $\mathbf{x}'$  and  $\tilde{\mathbf{x}}$ . Since  $\mathbf{p} \cdot \tilde{\mathbf{x}} \leq \mathbf{p} \cdot \hat{\mathbf{x}}$  and  $\mathbf{p} \cdot \mathbf{x}' < \mathbf{p} \cdot \hat{\mathbf{x}}$ , we have  $\mathbf{p} \cdot \mathbf{x}'' < \mathbf{p} \cdot \hat{\mathbf{x}}$ . But since  $\mathbf{x}'' \in \mathcal{N}$ , we also have  $u(\mathbf{x}'') > u(\hat{\mathbf{x}})$ . Thus,  $\hat{\mathbf{x}}$  does not minimize  $\mathbf{p} \cdot \mathbf{x}$  over the set  $\{x \in X \mid u(\mathbf{x}) \geq u(\hat{\mathbf{x}})\}$ , a contradiction. ■

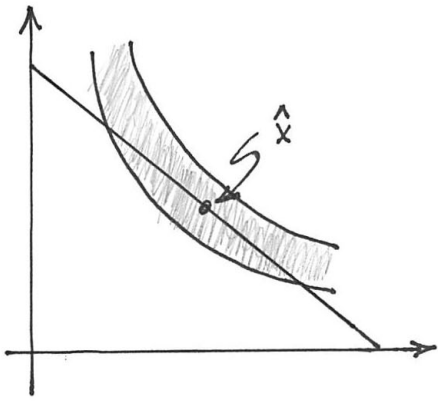


Figure 1

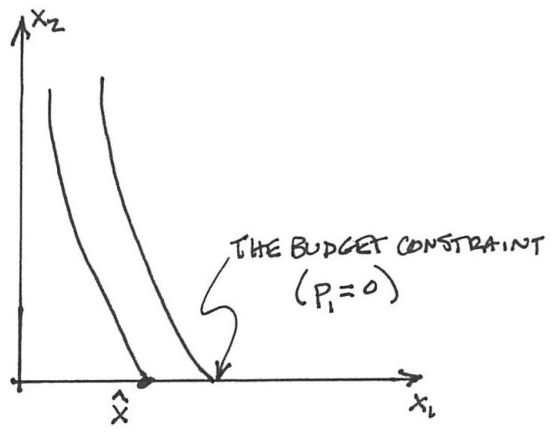


Figure 2

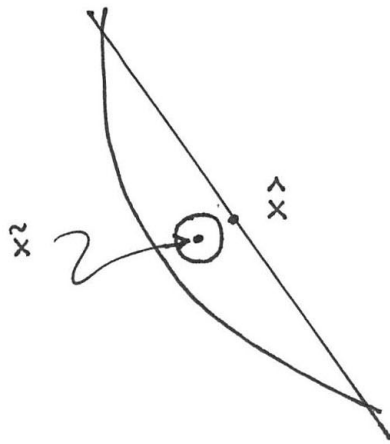


Figure 3

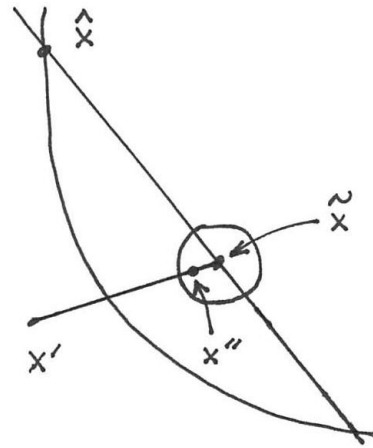


Figure 4